Exercise 2

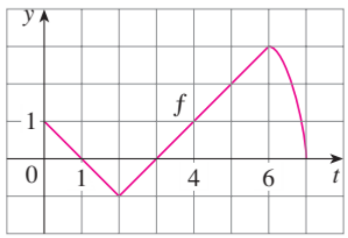
Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$, and 6.

(b) Estimate $g(7)$.

(c) Where does g have a maximum value? Where does it have a minimum value?

(d) Sketch a rough graph of g.



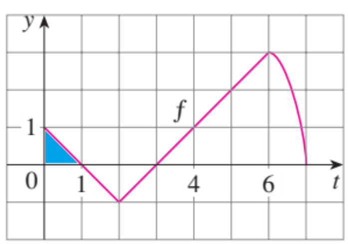
Solution

Part (a)

For $g(0)$, the limits of integration are the same, which makes the integral zero.

$$g(0) = \int_0^0 f(t)\,dt = 0$$

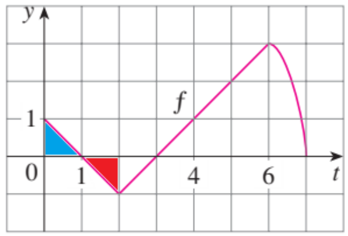
$g(1)$ is the area under the curve from $x = 0$ to $x = 1$.



The shape is a triangle with base 1 and height 1.

$$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}(1)(1) = \frac{1}{2}$$

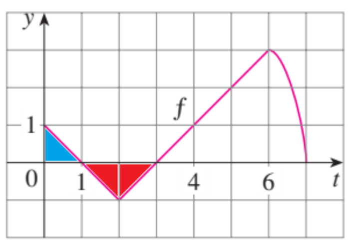
$g(2)$ is the area under the curve from $x = 0$ to $x = 2$.



The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$. The triangle shaded in red has base 1 and height -1, giving it negative area.

$$\begin{aligned} g(2) &= \int_0^2 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \\ &= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(-1) \\ &= 0 \end{aligned}$$

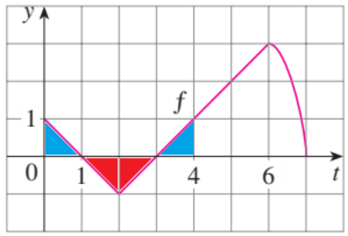
$g(3)$ is the area under the curve from $x = 0$ to $x = 3$.



The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$ and that from $x = 2$ to $x = 3$.

$$\begin{aligned} g(3) &= \int_0^3 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt \\ &= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(-1) + \frac{1}{2}(1)(-1) \\ &= -\frac{1}{2} \end{aligned}$$

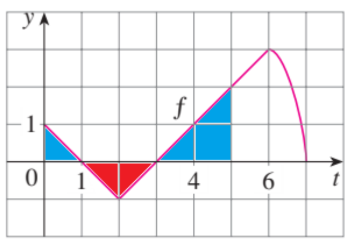
$g(4)$ is the area under the curve from $x = 0$ to $x = 4$.



The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$ and that from $x = 2$ to $x = 3$ and that from $x = 3$ to $x = 4$.

$$\begin{aligned} g(4) &= \int_0^4 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt \\ &= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(-1) + \frac{1}{2}(1)(-1) + \frac{1}{2}(1)(1) \\ &= 0 \end{aligned}$$

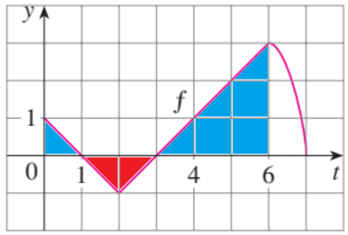
$g(5)$ is the area under the curve from $x = 0$ to $x = 5$.



The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$ and that from $x = 2$ to $x = 3$ and that from $x = 3$ to $x = 4$ and that from $x = 4$ to $x = 5$.

$$\begin{aligned} g(5) &= \int_0^5 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^5 f(t)\,dt \\ &= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(-1) + \frac{1}{2}(1)(-1) + \frac{1}{2}(1)(1) + \left[(1)(1) + \frac{1}{2}(1)(1)\right] \\ &= \frac{3}{2} \end{aligned}$$

$g(6)$ is the area under the curve from $x = 0$ to $x = 6$.

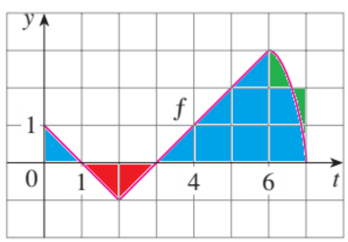


The area under the curve from $x = 0$ to $x = 1$ is added to that from $x = 1$ to $x = 2$ and that from $x = 2$ to $x = 3$ and that from $x = 3$ to $x = 4$ and that from $x = 4$ to $x = 5$ and that from $x = 5$ to $x = 6$.

$$\begin{aligned} g(6) &= \int_0^6 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^5 f(t)\,dt + \int_5^6 f(t)\,dt \\ &= \frac{1}{2}(1)(1) + \frac{1}{2}(1)(-1) + \frac{1}{2}(1)(-1) + \frac{1}{2}(1)(1) + \left[(1)(1) + \frac{1}{2}(1)(1)\right] + \left[(1)(1) + (1)(1) + \frac{1}{2}(1)(1)\right] \\ &= 4 \end{aligned}$$

Part (b)

$g(7)$ is the area under the curve from $x = 0$ to $x = 7$.



The area under the curve from $x = 6$ to $x = 7$ consists of two squares each missing a piece and a portion above them. The area of this portion looks roughly the same (perhaps slightly bigger) as that of the missing pieces.

$$\begin{aligned} g(7) &= \int_0^7 f(t)\,dt \\ &= \underbrace{\int_0^6 f(t)\,dt}_{=4} + \underbrace{\int_6^7 f(t)\,dt}_{\approx (1)(1) + (1)(1)} \\ &\approx 6 \end{aligned}$$

Part (c)

The minimum of g occurs at $x = 3$: $g(3) = -1/2$.
The maximum of g occurs at $x = 7$: $g(7) \approx 6$.

Part (d)

Below is a plot of $g(x)$ versus x for $0 \le x \le 7$.